My greatest hits — J. D. D. Martin — University of Waterloo

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Although I've never received anything like an encore, here I present — from sweet serenades to rambunctious rockers — my "greatest hits". Actually this is a selection of physics problems that I have developed for physics courses that I have taught at the University of Waterloo, ranging from second year optics to graduate quantum mechanics. They vary from utterly trivial to mildly challenging. Each problem has been selected for some unique aspect that I think makes it worth sharing. The randomness associated with the generation and selection of these problems obliges me to point out that they do not form any sort of coherent whole.

The problems are divided into subject areas, with brief introductory material. Use them in any way you see fit. Please let me know of any errors, improvements, etc... I'm happy to share my latex code. If you are stumped, or want to check your (or my) answer, please let me know. But don't ask me to send a solution without evidence that you have tried (or at least thought about) the problem.

I plan updates every few months or so. Enjoy!

Contents

1 Quantum mechanics

In the first lecture of my quantum mechanics course I discuss how matter wave diffraction experiments were able to conclusively demonstrate the existence of the weakly bound $^4He_2$ dimer; see W. Schöllkopf and J. P. Toennies, "The nondestructive detection of the helium dimer and trimer", J. Chem. Phys. **104**, 1155–1158 (1996). These are some of my "favourite experiments", and at some point in the past, I resolved that if I ever taught quantum mechanics, I would discuss them. In the following two problems based on this dimer, students get some exercise in one-dimensional quantum mechanics and numerical calculations. Links to the Blatt and Cooley papers can be found at the end of this collection in the References section [2–4].

(635) At

https://gist.github.com/jddmartin/7d1b237e64352582a67a36a6384357f7

you will find a Python function that computes the effective potential energy between two helium atoms $V(r)$ as a function of their internuclear separation r. The Born-Oppenheimer approximation (based on the large mass difference between nuclei and electrons) allows one to treat the relative motion of the two helium atoms using this potential, ignoring the specifics of the electronic degrees of freedom.

As with the hydrogen atom, we can apply separation of variables in spherical coordinates to the coordinates describing the internuclear separation, to obtain: $\psi(r, \theta, \phi) = R(r) Y_{\ell,m}(\theta, \phi)$ with $Y_{\ell,m}(\theta, \phi)$ being the standard spherical harmonics and $u(r) = rR(r)$ satisfying the one-dimensional Schrödinger equation:

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}\right] u(r) = Eu(r)$$

where μ is the *reduced* mass of the two bodies.

The existence of bound states can be determined by using the Numerov algorithm to solve for $u(r)$ with the energy $E = 0$. You should start at low-r, integrating in the outwards direction to large enough r such that $u(r)$ is a reasonable approximation to a straight line (the appendix of the paper of Cashion provides a succinct summary of the Numerov technique, but Blatt or Cooley should be consulted for its derivation; these papers are posted on the web-site).

(a) For $E = 0$ provide plots for $u(r)$ versus r with $\ell = 0$ for $^3He_2$ and $^4He_2$ (out to the straight line regime) and $\ell = 1$ for $^4He_2$ (to the same distances as for $\ell = 0$). (Use the provided $V(r)$ – converted to the computer language of your choice).

(b) For the three cases in the previous part, determine whether or not bound states with $E < 0$ will form (explain your reasoning). Hint: examine the different cases shown in Figure 6.12 of Sakurai.

(c) For the $\ell = 0$ cases, report the distance r at which the asymptotic straight line intersects with the horizontal r axis. (This is known as the *scattering length* — its relevance to low-energy scattering will be discussed later in the course.)

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(638) In the previous assignment you integrated Schrödinger's equation (with $E = 0$) for the He_2 potential, starting at low-r (internuclear separation), out to large-r using Numerov's method. In the current assignment you will determine the bound state energy eigenvalues of this system (when measured relative to $V(\infty)$ these are the "binding energies").

The basic idea is to integrate outwards as you have done, but also to perform a similar integration in the opposite direction (starting at large-r and integrating towards low-r). At a point where the wavefunctions meet (normally chosen to be the outer classical turning point), you will be able to rescale each solution so that they have the same values at this point. But unless the energy used to integrate is an energy eigenvalue, there will be a discontinuity in the slope of the wavefunction at the meeting point.

Cooley developed a method for using this discontinuity to obtain a "better" value of the energy which will reduce this discontinuity. The inwards and outwards integrations can be redone using this energy and a new energy estimate obtained from the discontinuity in the same way. Thus energy eigenvalue estimation becomes an iterative procedure, each step providing a better estimate for an energy eigenvalue (with remarkably fast convergence compared to any "naive" method). (Again Cashion gives a succinct description of the technique – Blatt provides a more detailed explanation; these papers are posted to Learn.)

(a) To check your implementation of Cooley's method you can determine the binding energy of $^4He_2$ (in eV) and compare your binding energy to that found in Table I of Janzen and Aziz, J. Chem. Phys. **103**, 9626 (1995): $E_{\text{bind}} = 1.594\,\text{mK}$ for the `HFD-B3-FCI1` potential. Normally the lowest energy "vibrational" wavefunction for a diatomic molecule (e.g. H_2) looks very similar to the $n = 0$ harmonic oscillator wavefunction. For this unusual weakly bound system, you should find that it looks quite different.

(b) Using the same approach as for $^4He_2$ determine the binding energy of the lowest vibrational state of $^6He^6He$ (in eV), and estimate the uncertainty in the binding energy that you have obtained. (^6He is a short-lived radioisotope, but it should be possible *in principle* to observe bound states of this system — whereas we can determine from the approach of the first assignment that $^3He^3He$ cannot exist.)

(c) Provide a plot comparing the normalized wave functions for the preceeding two parts (label the axes appropriately, including units).

Napolitano added a nice example of diffractive scattering to the most recent edition of Sakurai's text [5]. Interestingly, this data also provides a nice illustration of the Rutherford formula and its breakdown:

(476) In the lectures we discussed the Calcium-proton scattering experiments of Ray *et al.*, Phys. Rev. C, **23** 828 (1981). In particular, we found that a simple finite square-well model could predict the angles of the diffraction minima in $d\sigma/d\Omega$. However the experimental $d\sigma/d\Omega$'s do not go all the way to zero at these angles (see Fig. 1 of Ray *et al.*). Use the Rutherford scattering formula to estimate numerical values for $d\sigma/d\Omega$ at these minima and comment on the possible reasons for any discrepencies.

Casimir's original paper on [6] "his effect" is surprisingly readable, suggesting:

(646) (a) In the lectures I outlined Casimir's derivation of the attractive force F between two conducting plates of area A due to zero-point energy:

$$\frac{F(a)}{A} = \frac{\pi^2}{240}\frac{\hbar c}{a^4}$$

where a is their separation.

Casimir's result is for three spatial dimensions (so-called "3+1"). Derive the analogous result (attraction between two "point conductors"?) due to the zero point energy of a scalar field in one spatial dimension ("1+1"):

$$F = \frac{\pi}{24}\frac{\hbar c}{a^2}$$

I recommend working this out following Casimir's original approach and not googling for the answer (Ambjørn and Wolfram — yes, that Wolfram! — are among the many people who have looked at the problem in arbitrary spatial dimensions http://dx.doi.org/dkg763). I have posted a copy of Casimir's original paper to the course web-site.

(b) Outline some (possible) applications of the Casimir effect in "d+1" where $d < 3$ or $d > 3$. (On the order of a paragraph.)

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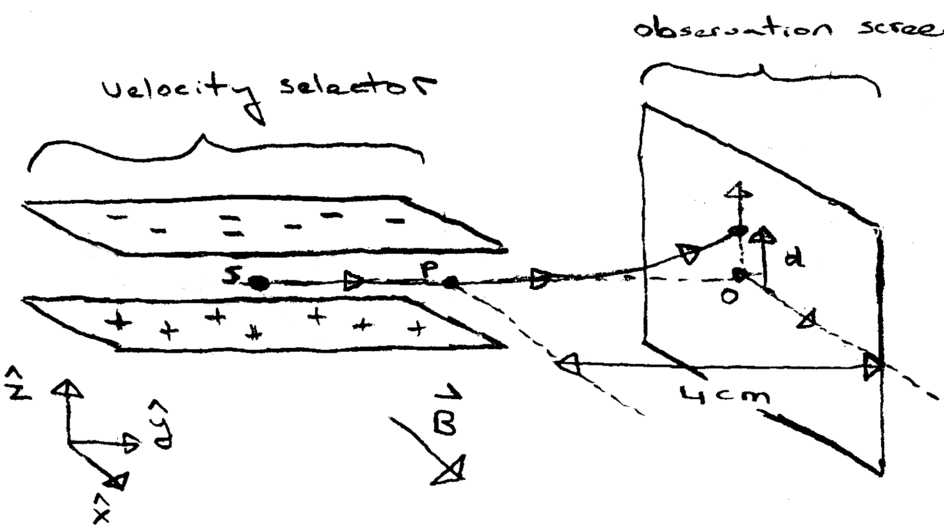
2 Special relativity

In Winter 2019, I taught Phys 263, *Classical mechanics and special relativity* for the first time. Often in a course like this, special relativity gets the proverbial short end of the stick. I was determined not to let that happen, and thus started the course with relativistic kinematics. Classical (non-relativistic) dynamics was covered around the middle of the course, and the course finished with relativistic dynamics. I liked this approach and recommend it. I wanted to show as many experimental aspects of relativity as possible, as reflected in the following questions.

I obtained the idea and numbers for the following problem from Williams, *Introducing Special Relativity* [7], a book that deserves to be better known and more widely available. It would be nice to follow this question up with another on the limitations on cyclotrons due to special relativity.

(695) When special relativity was being developed it was not straightforward to accelerate charged particles to velocities close enough to the speed of light for the corrections to Newtonian mechanics to be observable. Instead, relativistic electrons due to beta decay radioactivity were used. However, these sources emit electrons over a *range* of velocities — necessitating *velocity-selection*, which we have discussed earlier in this course.

Here is a simplified version of an apparatus due to A. Bucherer that provided early experimental evidence for the laws of relativistic mechanics:



The apparatus operates according to the following principles:

(i) A radioactive source S emits electrons with a range of velocities.

(ii) The combination of an electric field $\vec{E} = E_z\hat{z}$ (created using two charged metal plates above and below the source) and magnetic field $\vec{B} = B_x\hat{x}$ act as a velocity-selector, so that electrons that arrive at P have a well-defined velocities (the plates are placed close together, so that electrons of the wrong velocities hit the plates and do not make it to the observation screen).

(iii) Once the velocity-selected electrons move outside of the region between the two plates (past P), the electric field drops to zero quickly, whereas the magnetic field $\vec{B} = B_x\hat{x}$ remains constant.

(iv) If the velocity-selected electrons were to remain undeflected past P they would arrive at point O on the observation screen, a distance 4 cm away from the exit of the selector. Instead, the electrons are deflected upwards once they leave the selector due to $\vec{B}$ and arrive a distance d upwards from O. Distances d may be measured for different values of v/c (by changing the velocity-selector fields).

The amount that the electrons are expected to travel upwards (d) differs depending on whether it is computed using relativistic or non-relativistic (Newtonian) dynamics. Thus precisely measuring d provides a test of special relativity. The test becomes more stringent the closer the electron's velocities are to c.

Questions:

(a) If $E_z = 2 \times 10^6\,\mathrm{V/m}$ what must B_x be to select electrons with $v/c \approx 0.7$?

(b) If $v/c \approx 0.7$, what is d according to non-relativistic dynamics? (Call it $d_{\text{non-rel}}$.)

(c) If $v/c \approx 0.7$, what is d according to relativistic dynamics? (Call it d_{rel}.)

(d) In practice the precise location of the point O is not straightforward to determine, contributing to uncertainty in the measurement of d. How should $\vec{E}$ and $\vec{B}$ be changed so that the electrons are deflected *downwards* from O? How would this change help to measure d?

A wonderful illustration of relativistic time dilation is the Mount Washington muon decay experiment — known more generally as the "Rossi-Hall" experiment — as described in: D. H. Frisch and J. H. Smith, "Measurement of the Relativistic Time Dilation Using mu-Mesons", Am. J. Phys. **31**, 342–355 (1963)

There is an accompanying movie: `https://youtu.be/tbsdrHlLfVQ` It is a bit old-fashioned and slow-paced, but I enjoyed it. Do you need Mount Washington to do the experiment?

(658) Consider the test of relativistic time dilation with muons discussed in the lectures. This test depended on comparing muon count rates at sea-level with those on top of Mount Washington (1917 m above sea-level). We do not have any large mountain nearby Kitchener-Waterloo to do this test. However, it is possible to go from 75 m above sea-level (Lake Ontario), to about 250 m above sea-level by travelling about 15 km up the Dundas Valley in Hamilton. Suppose that like in the case of the Mount Washington experiment, we have a muon detector sensitive to muons at $0.9952c$. Would we be able to test relativistic time-dilation by comparing muon count rates between these two elevations? Make sure to include estimated quantities to support your argument for or against the feasibility of this experiment.

I started the course with Mermin's derivation of the relativistic velocity addition formula [9] (which I recommend), and thus also discussed the related Fizeau experiment, prompting the following question.

(657) (a) The test of the relativistic velocity addition formula discussed in the lectures is known as the *Fizeau "aether-drag" experiment*. This experiment measures small phase shift differences $\ll 2\pi$ between the two paths (CCW vs CW). To see why only small differences are practical, compute the water speed required to change the phase difference between the two paths by π, for $L = 4\,\mathrm{m}$, $\lambda_{\mathrm{air}} = 532\,\mathrm{nm}$, and $n = 1.33$ (as used in the experiment described by Lahaye *et al.*). What practical problems might arise with this water speed?

(b) On the final exam or midterm you may be asked to explain the Fizeau aether-drag experiment and derive the relevant formulae for the change in the phase shift difference in both the relativistic and non-relativistic cases. If you have any remaining questions on this derivation, please note them here, and I will attempt to address them. If you do not have any questions, you do not have to write anything.

3 Classical mechanics

I was fortunate that just prior to the first time that I taught Phys 263, New Horizons was in the news. The following problem was inspired by Ref. [10].

(653) Around January 1st, 2019, the New Horizons spacecraft passed by and imaged "Ultima Thule". Earlier (in 2015) New Horizons passed by Pluto, sending beautiful images back to earth. To get to Pluto (and Ultima Thule) quicker, New Horizons used a Jupiter gravity assist (in a similar manner as Pioneer 10). NASA stated that *"New Horizons will reach the Pluto system in July 2015 – five years earlier than without the Jupiter boost."*

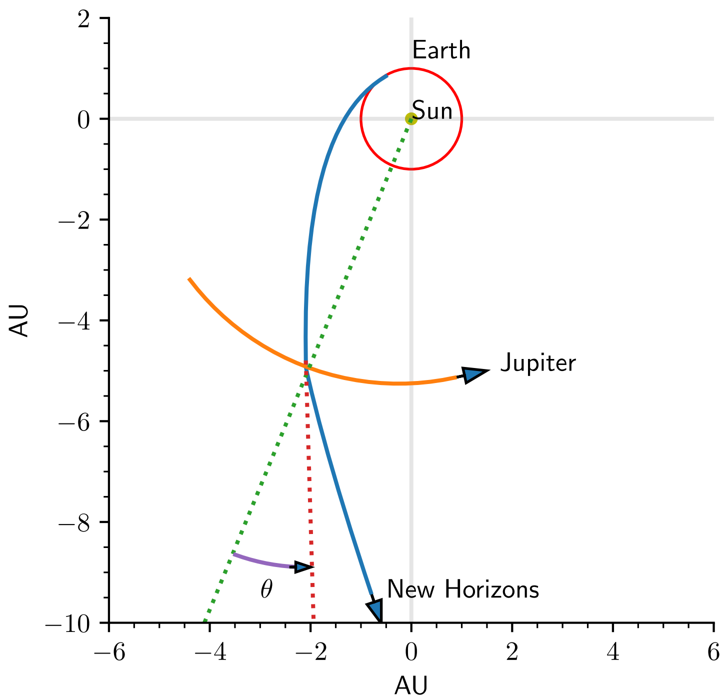
Immediately prior to the gravity assist, the New Horizons spacecraft was moving at a speed of $19.5\,\mathrm{km/s}$ in the sun-fixed frame. The spacecraft was moving in a direction $\theta = 24.2°$ counterclockwise from the line pointing from the sun to Jupiter (see diagram below).

Due to the gravity assist, the spacecraft's trajectory was deflected by $\beta = 15.7°$ (counterclockwise) *in the Jupiter-fixed frame*.

In the sun-fixed frame, Jupiter's speed at the encounter was $12.7\,\mathrm{km/s}$.

Compute the speed of the New Horizons spacecraft after its encounter with Jupiter, in the sun-fixed frame.

(A useful plot for checking your answer can be found on the wikipedia page for New Horizons.)



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4 Electricity and magnetism

For many years the standard textbook for graduate electricity and magnetism was Jackson's *Classical Electrodynamics* [11], which discusses the "exactness" of the $1/r^2$ behaviour in electrostatics, suggesting the following problem:

(485) (a) Consider Maxwell's experimental test of Coulomb's law:

(i) Start with two concentric spherical conducting shells (outer radius a, inner radius b) insulated from one another, initially uncharged.

(ii) Charge the outer shell up to potential V_{test} with respect to infinity (ground).

(iii) Remove the source of potential, and put the inner and outer shells into electrical contact.

(iv) Break contact between the shells, ground the outer shell, and measure the potential of the inner shell with respect to ground. Call this V_α.

Explain why V_α should be zero if Coulomb's law is exact.

(b) Consider a modified form of Coulomb's law, in which the field due to a point particle of charge q is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^{2+\alpha}}\hat{\mathbf{r}} \tag{1}$$

where $|\alpha| \ll 1$. Show that for the modified form of Coulomb's law (Eq. 1), the field *within* a uniformly charged thin spherical shell with surface charge σ and radius R is (to first order in α):

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0 (r/R)^2}\left[2(r/R) + \ln\left(\frac{1-(r/R)}{1+(r/R)}\right)\right]\alpha\,\hat{\mathbf{r}}$$

where r is the distance from the center of the spherical shell and $\hat{\mathbf{r}}$ is the unit vector pointing out from its center.

(c) With $\alpha \neq 0$ the value of V_α in Maxwell's test will no longer be zero. Derive the following relationship between a, b, V_{test}, α and V_α, that can be used to quantify possible deviations from Coulomb's law (to first order in α):

$$V_\alpha = \frac{V_{\text{test}}\alpha}{2}\left[n \ln\frac{n+1}{n-1} - \ln\frac{4n^2}{n^2-1}\right]$$

where $n = a/b$, the ratio of the outer to inner radii of the two spherical shells.
[The following integral may be useful:

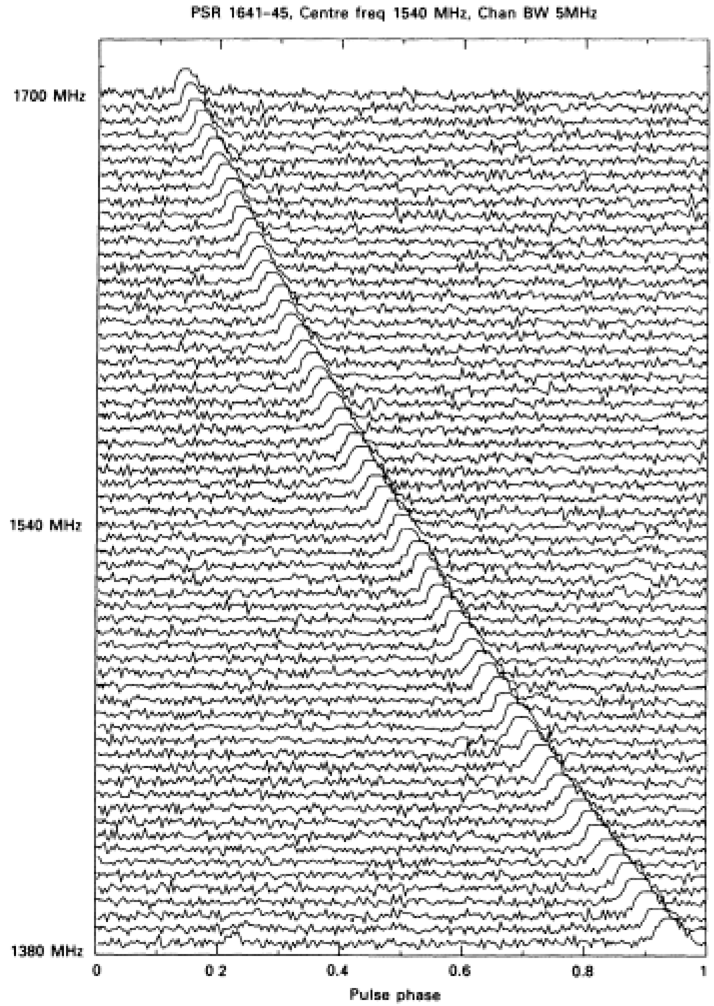
$$\int \frac{1}{x^2}\ln(1-x)\,dx = \frac{-1}{x}\ln(1-x) + \ln\frac{1-x}{x} + C.$$

]

(d) Consider an experimental setup with $V_{\text{test}} = 3000$ V, $a = 2.5$ ft and $b = 2.0$ ft. If the smallest measurable $|V_\alpha|$ is 1 μV, what upper bound can be placed on $|\alpha|$?

Pulsars emit short pulses of electromagnetic radiation that are lengthened in duration as they travel through a dispersive media. The associated calculations nicely complement the treatment of dispersion in Zangwill's *Modern Electrodynamics* [12]. And of course "chirped-pulses" have a special place in the heart of any physicist at Waterloo.

(548) Shown below are observations of a "chirped" radio-frequency pulse due to Pulsar 1641-45 (Figure 16.20 of Carroll and Ostlie, Introduction to Modern Astrophysics, 2nd ed.). Pulses are observed from this pulsar every 0.455 s.



(a) The observed chirp is caused by the group velocity dispersion of the interstellar medium (ISM), and is primarily due to free electrons. Using Eq.' s 18.92 and 18.107 of Zangwill, show that the product of the ISM electron number density n and distance from earth to the pulsar z can be estimated using observations of $\omega^3/(d\omega/dt)$ (and some well-known physical constants). The quantity nz is known in pulsar astronomy as the "Dispersion Measure". You may assume $\omega \gg \omega_p$. [Please note that there is a slight typo in Zangwill's Eq. 18.107. The Δ that appears in the denominator should be ΔT.]

(b) Using the experimental observations from the figure above, estimate nz. Express your result in units of m^{-2} and $\mathrm{cm}^{-3} \times \mathrm{pc}$ (where pc means a "parsec").

(c) Pulsar 1641-45 has been estimated (by independent means) to be $3.9 - 5.3\,\mathrm{kpc}$ away. Based on your answer to the previous part, estimate the average electron density n of the ISM.

(d) Confirm the approximation $\omega \gg \omega_p$. Why can ions in the ISM be ignored when computing the group velocity dispersion?

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5 Optics

Check back soon ...

6 Thermodynamics

Most recently, I have been teaching third year thermodynamics, using Schroeder's text [13], which contains many wonderful examples, and helped inspire the following problems.

(414) The operation of hot-air balloons (Schroeder's Figure 1.1) can be explained using Archimedes' principle. The air inside the large (normally colourful) envelope is heated and thus less dense than the surrounding air.

The envelope air temperature is limited to temperatures of less than 120 °C to prevent degradation of the envelope material (normally nylon). However, balloon performance is normally calculated using envelope air temperatures of 100 °C. This gives some reserve capability to rapidly ascend by temporarily increasing the temperature of the air in the envelope (to no more than 120 °C).

Consider a single-person "hopper balloon" with an envelope volume of 400 m^3, a 100 °C envelope air operating temperature, operating at sea-level with a surrounding air temperature of 20 °C.

(a) Compute the maximum payload for the balloon (in kg) (do not include the mass of the air within the envelope as part of the payload).

(b) Compute the mass of the air within the envelope (in kg).

(c) How would the allowable payload mass change due to:

(i) lower surrounding air temperatures?

(ii) elevations higher than sea level?

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(614) (a) How much force (in N) is required to keep a 16 g ice cube completely submerged under water?

(b) A 660 g stainless steel pan containing 1000 ml of water is placed on a stove ring. Both the pan and water are initially at room temperature. After approximately 8 minutes, the water begins to boil vigorously. The pan is left on the stove. *Estimate* how much longer (in minutes) it will take the pan to boil dry (i.e. no water is left in pan). State your assumptions. (Stainless steel has a specific heat capacity of $c_P \approx 500\,\mathrm{J/(kg \cdot K)}$).

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